2.6.3 Changing Money Jesus went into the temple, and began to cast out them that sold and bought in the temple, and overthrew the tables of the moneychangers ... — Mark 11:15

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You use the Maclaurin series for the exchange of coins

------------------------------------------------------------------------------------------------------------------------------------------2.6.4 Fibonacci Numbers Attention!

Attention! Ladies and gentlemen, attention! There is a herd of killer rabbits headed this way and we desperately need your help! — Night of the Lepus

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Colonies of rabbid killer rabbits growing.

2.6.5 Recurrence Relations

O me! O life!. . . of the questions of these recurring;

— Walt Whitman, Leaves of Grass

Its basic

2.6.6 Catalan Numbers

zero, un, dos, tres, quatre, cinc, sis, set, vuit, nou, deu, onze, dotze, tretze, catorze, quinze, setze, disset, divuit, dinou, vint.

How many ways to compute k+1 matrices (matrix multiplication is associative but not commutative)

We use the Maclaurin serires to find the catalan numbers

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Sloane and Plouffe [258] remark that the Catalan numbers are perhaps the second most frequently occurring numbers in combinatorics, after the binomial coefficients. Indeed, Stanley [262, ex. 6.19] lists 66 different combinatorial interpretations of these numbers! We close with another problem whose solution involves the Catalan numbers.

A rooted tree is a tree with a distinguished vertex called the root. The vertices in a rooted tree form a hierarchy, with the root at the highest level, and the level of every other vertex determined by its distance from the root. Some familiar terms are often used to describe relationships between vertices in a rooted tree: If v and w are adjacent vertices and v lies closer to the root than w, then v is the parent of w, and w is a child of v. Likewise, one may define siblings, grandparents, cousins, and other family relationships in a rooted tree.

We say that a rooted tree is strictly binary if every parent vertex has exactly two children. How many strictly binary trees are there with k parent vertices? Do not take symmetry into account: If two trees are mirror images of one another, count both configurations. Figure 2.4 shows that there are five trees with three parent vertices.

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2.7 Polya’s Theory of counting

Who are you who are so wise in the ways of science?

— Sir Bedivere, in Monty Python and the Holy Grail

How many ways can King Arthur and his knights sit at the round table? How many different necklaces with n beads can be formed using m different kinds of beads?

This is like the exchanging of coins to create different configurations.

2.7.1 Permutation Groups I haven’t fought just one person in a long time. I’ve been specializing in groups. — Fezzik, in The Princess Bride

A group consists of a set G together with a binary operator ◦ defined on this set. The set and the operator must satisfy four properties.

• Closure. For every a and b in G, a ◦ b is in G.

• Associativity. For every a, b, and c in G, a ◦ (b ◦ c)=(a ◦ b) ◦ c

• Identity. There exists an element e in G that satisfies e ◦ a = a ◦ e = a for every a in G. The element e is called the identity of G.

• Inverses. For every element a in G, there exists an element b in G such that a ◦ b = b ◦ a = e. The element b is called the inverse of a.

In addition, if a ◦ b = b ◦ a for every a and b in G, we say that G is an abelian, or commutative, group.

STUDY ABSTRACT ALGEBRA

The Cyclic Group

If π is a permutation in Sn and m is a nonnegative integer, let πm denote the permutation obtained by composing π with itself m times, so π0 = (1), and π3 = π ◦ π ◦ π.

Let π = {πm : m ≥ 0},

Again study abstract algebra

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The Dihedral Group

The dihedral group Dn is the group of symmetries of a regular polygon with n sides, including reflections as well as rotations. Since Cn consists of just the rotational symmetries of such a figure, evidently Cn is a subgroup of Dn.

2.7.2 Burnside’s Lemma

Burnside had submitted the scheme to Meade and myself, and we both approved of it, as a means of keeping the men occupied. — Personal Memoirs of U. S. Grant

• reflexive: c ∼ c for all colorings c,

• symmetric: c1 ∼ c2 implies c2 ∼ c1, and

• transitive: c1 ∼ c2 and c2 ∼ c3 implies c1 ∼ c3.

Lemma 2.8. Suppose a group G acts on a set of colorings C. For any coloring c in C, we have |Gc| | c | = |G|.

Theorem 2.9 (Burnside’s Lemma).

The number of equivalence classes N of the set C in the presence of the group of symmetries G is given by N = { 1 / |G| }

π∈G

2.7.3 The Cycle Index

Lance Armstrong (7), Jacques Anquetil (5), Bernard Hinault (5), Miguel Indurain (5), Eddy Merckx (5), Louision Bobet (3), Greg LeMond (3), Philippe Thys (3).

— Multiple Tour de France winners

To use Burnside’s Lemma to count the number of equivalence classes of a set of colorings C, we must compute the size of the invariant set Cπ associated with every permutation π in a group of symmetries G. A simple observation allows us to compute the size of this set easily in many situations.

Suppose we wish to determine the number of ways to color n objects using up to m colors, discounting symmetries on the objects described by a group G. If a coloring is invariant under the action of a permutation π in G, then every object permuted by one cycle of π must have the same color.

2.7.4 Polya’s Enumeration Formula

´ I have yet to see any problem, however complicated, which, when looked at in the right way, did not become still more complicated. — Poul Anderson

We can use the cycle index to solve more complicated problems on arrangements in the presence of symmetry. Suppose we need to determine the number of equivalence classes of colorings of n objects using the m colors y1, y2,..., ym, where each color yi occurs a prescribed number of times. For example, how many different necklaces can be made using exactly two rhodonite, nine rose quartz, and nine lapis lazuli beads?

Theorem 2.10 (P´olya’s Enumeration Formula).

Suppose S is a set of n objects and G is a subgroup of the symmetric group Sn. Let PG(x) be the cycle index of G. Then the pattern inventory for the nonequivalent colorings of S under the action of G using colors y1, y2,..., ym is m m m

Fg(y) = Pg ( , ∑yi^2 , … , ∑yi^n )

I=1 i=1 i=1

2.7.5 de Bruijn’s Generalization

It doesn’t matter what color, well that gets a nope! Be it pink, purple, or heliotrope!

— Boundin’, Pixar Films

Theorem 2.11. Suppose S is a set of n objects, R = {y1,...,ym} is a set of m colors, G is a subgroup of the symmetric group Sn, and ρ ∈ Sm. Let PG(x) denote the cycle index of G. Then the pattern inventory for the colorings of S which are nonequivalent with respect to the action of G on S, but invariant with respect to the action of ρ on R, is FG,ρ(y) = PG(α1(ρ), α2(ρ),...,αn(ρ)),

Where k-1

αk(ρ) = ∑ ∏ yp^i(j)

p^k(j)=j i=0

Theorem 2.12 (de Bruijn’s Enumeration Formula). Suppose S is a set of n objects, R = {y1,...,ym} is a set of m colors, G is a subgroup of the symmetric group Sn, and H is a subgroup of Sm. Then the pattern inventory FˆG,H(y) for the colorings of S which are nonequivalent with respect to both the action of G on S and the action of H on R is obtained by identifying equivalent color patterns in the polynomial

F G, H (y) = 1 ∑ F G, p (y)

* p

|H|

Where F G, p (y) is given by (2.64).

Corollary 2.13.

Suppose S is a set of n objects, R is a set of m colors, G is a subgroup of the symmetric group Sn, and H is a subgroup of Sm. Then the number of colorings of S using the colors in R which are nonequivalent with respect to both the action of G on S and the action of H on R is

N G, H (n, m) = 1

\_ ∑ P G (,

|H| p

Where = ∑ i/k j, with the sum extending over all the positive divisors j of k and is the number of cycles of p of length j.